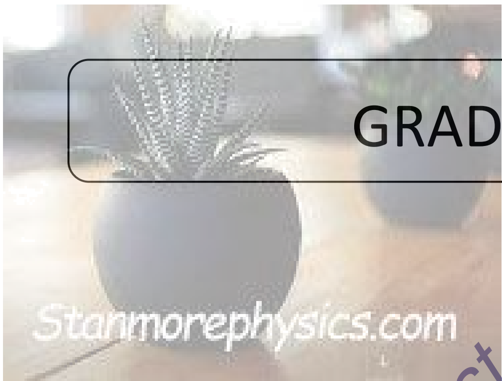
FRANS DU TOIT

VIRIBUS EXCELSIOR



GRADE 12

MATHEMATICS

INVESTIGATION

MARKS: 50

IN THIS INVESTIGATION YOU WILL :

* Learn terminology and symbols that you can use in sequences and series.
* Guided step for step in solving typical problems.
* Tested with each problem.

QUESTION 1

1.1 Consider the series $2 + 5 + 8 + 11 + 14 + 17$.

Given: $S_n = T_1 + T_2 + T_3 + \cdots + T_n$

$\therefore S_4 = T_1 + T_2 + T_3 + T_4$

Therefore: $S_1 = 2$ en $S_2 = 2 + 5 = 7$

Complete:

1.1.1 $S_3 =$ ___ (1)

1.1.2 $S_4 =$ ___ (1)

1.2 Calculate the sum of the first 4 terms of the series with $T_k = 2k - 1$. (1)

$S_n = [2(\_) - 1] + [2(\_) - 1] + [2(\_) - 1] + [2(\_) - 1]$

$= $ ____

S_n can also be written as:

$$\sum_{k=1}^{4} 2k - 1 = [2(1) - 1] + [2(2) - 1] + [2(3) - 1] + [2(4) - 1]$$

1.3 Calculate the following: (2)

$$\sum_{r=0}^{5} 3 \cdot 2^r$$

1.4 How many terms is in:

1.4.1 the series in 1.3? (1)

1.4.2 $\sum_{k=1}^{8} 3k$ (1)

1.4.3 $\sum_{k=3}^{8} 3k$ (1)

1.5 Give a general expression for the number of terms in the series: (1)

$$\sum_{k=m}^{n} 3k$$

[9]

QUESTION 2: To write a series in sigma notation

2.1 Consider the series $1 + 5 + 9 + \cdots + 97$

2.1.1 Calculate the general term of the series. (2)

2.1.2 Calculate how many terms is in the series. (2)

2.1.3 Write the series in sigma notation. (1)

2.2 Write the series $(-3) + (-1) + 3 + 9 + \cdots$ to n terms in sigma notation. (3)

[8]

QUESTION 3: The sum of arithmetic series.

Given:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

3.1 To calculate the sum of the first 20 terms of the series $2 + 5 + 8 + 11 + \cdots$:

3.1.1 Give the values of a, n and d.. (1)

3.1.2 Calculate now the sum of the first 20 terms. (1)

3.2 Consider the series $2 + 5 + 8 + \cdots + 158$

3.2.1 Determine the general term T_n of the series. (1)

3.2.2 Calculate the number of terms in the series. (1)

3.2.3 Hence, determine the sum of the series. (2)

3.3 To calculate how many terms of the arithmetic row, 3; 5; 7; ...... must be added to be equal to 440:

3.3.1 Give the values of a and d. (1)

3.3.2 Hence, determine the value for n. (2)

3.4 A supermarket stacks the cold drink bottles in a triangle on top of each other. One bottle at the top, two bottles in the second row, three bottles in the third row and so on. A group of campers enter to buy the first 5 rows of bottles. How many bottles are left if the last row had 20 bottles? (3)

[12]

QUESTION 4: The sum of geometric series.

Given:

$$T_n = ar^{n-1} \qquad S_n = \frac{a(r^n - 1)}{r - 1}\ ; r \neq 1 \qquad \text{where} \quad r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

4.1 Determine the sum of the first 15 terms of the series: (2)

$$\frac{2}{3} + 2 + 6 + \cdots$$

4.2 Consider the ending series: $256 + 128 + 64 + 32 + \cdots + 0.25$

4.2.1 Give a value for a and r and hence T_n. (1)

4.2.2 Calculate n, the number of terms in the series. (1)

4.2.3 Hence determine the sum of the ending series. (2)

4.3 To calculate m .

$$\sum_{k=3}^{m} \frac{1}{16}(2)^k = 127.5$$

4.3.1 Calculate a and r. (1)

4.3.2 Determine the number of terms. (1)

4.3.3 Calculate m. (3)

[11]

QUESTION 5: The sum to infinity of a convergent geometric series.

5.1 Choose the correct option between the brackets:

5.1.1 When a geometric series converge, each consecutive term must be **(smaller/bigger)** than the previous one. (1)

5.1.2 Therefore $(\boldsymbol{r} > \pm 1 \ ; \ -\mathbf{1} < r < 1)$ (1)

5.2 Given:

$$S_\infty = \frac{a}{1-r}$$

5.2.1 The second term of a convergent series is $\frac{5}{2}$ and the sum to infinity is 10. Calculate the constant ratio. (4)

5.3 Given: $0.\dot{2}\dot{3}$ (Write as a normal fraction)

5.3.1 Write the decimal as a geometric series. (1)

5.3.2 Calculate a and r. (1)

5.3.2 Calculate the sum to infinity. (2)

[10]

Total: [50]

MEMORANDUM

Grade 12

INVESTIGATION 1

QUESTION 1

1.1 1.1.1 $S_3 = 7 + 8 = 15$ (1)

1.1.2 $S_4 = 15 + 11 = 26$ (1)

1.2 $T_k = 2k - 1.$ (1)

$S_n = [2(1) - 1] + [2(2) - 1] + [2(3) - 1] + [2(4) - 1]$

$= 1 + 3 + 5 + 7$

$= 16$

1.3 (2)

$$\sum_{r=0}^{5} 3 \cdot 2^r = 3 \cdot 2^0 + 3 \cdot 2^1 + 3 \cdot 2^2 + 3 \cdot 2^3 + 3 \cdot 2^4 + 3 \cdot 2^5$$

$= 3 + 6 + 12 + 24 + 48 + 96$

$= 189$

1.4 The number of terms in:

1.4.1 the series in 1.4 is 6 (1)

1.4.2 $\sum_{k=1}^{8} 3k$ is 8 (1)

1.4.3 $\sum_{k=3}^{8} 3k$ is 6 (1)

1.5 The number of terms in the series: (1)

$$\sum_{k=m}^{n} 3k \qquad is\ n - (m - 1) = n - m + 1$$

[9]

QUESTION 2: To write the series in sigma-notation

2.1 Given: $1 + 5 + 9 + \cdots + 97$

2.1.1 The first difference is constant. ⟶ Arithmetic series.

General term: $T_n = a + (n - 1)d$

$a = 1; \quad d = 4$

$\therefore T_n = 1 + (n - 1)4$

$= 4n - 3$ (2)

2.1.2 The last term is $4n - 3 = 97$

$$\therefore \quad 4n = 97 + 3$$

$$\therefore \quad n = \frac{100}{4}$$

$$= 25$$ (2)

2.1.3 In sigma notasion: (1)

$$\sum_{n=1}^{25} 4n - 3$$

2.2 $(-3) + (-1) + 3 + 9 + \cdots$ (3)

$$+2 \quad +4 \quad +6$$

$$+2 \quad +2$$

Second difference constant:⟶ Quadratic number pattern $\therefore T_n = an^2 + bn + c$

$2a = 2 \qquad \therefore a = 1$

$3a + b = 2 \qquad \therefore b = 2 - 3 = -1$

$a + b + c = -3 \qquad \therefore 1 - 1 + c = -3 \qquad \therefore c = -3$

$\therefore T_n = n^2 - n - 3$

$$\therefore \sum_{k=1}^{n} k^2 - k - 3$$

[8]

QUESTION 3: The sum of arithmetic series:

3.1 To calculate the sum of the first 20 terms of the series $2 + 5 + 8 + 11 + \cdots$:

3.1.1 $a = 2$, $n = 20$ and $d = 3$.. (1)

3.1.2 $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{20} = \frac{20}{2}[2(2) + (20 - 1)3]$$

$$= 10(61)$$

$$= 610$$ (1)

3.2 Given: $2 + 5 + 8 + \cdots + 158$

3.2.1 $T_n = 2 + (n-1)3 = 3n - 1$ (1)

3.2.2 $3n - 1 = 158$

$\therefore \quad 3n = 159$

$\therefore \quad n = 53$ (1)

3.2.3

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$= \frac{53}{2}[2(2) + (53-1)3]$$
$$= 4240$$
(2)

3.3 To calculate the number of terms in the arithmetic row that will add up to 440: $3; 5; 7; \ldots$

3.3.1 $a = 3$ and $d = 2$. (1)

3.3.2

$$\frac{n}{2}[2a + (n-1)d] = 440$$
$$\therefore \frac{n}{2}[2(3) + 2(n-1)] = 440$$
$$\therefore \quad n[6 + 2n - 2] = 880$$
$$\therefore \quad 2n^2 + 4n - 880 = 0$$
$$\therefore \quad n^2 + 2n - 440 = 0$$
$$\therefore \quad (n+22)(n-20) = 0$$
$$\therefore \quad n = 20 \qquad n \neq -22$$
(2)

3.4 The general term of the row is $T_n = n$

The number of bottles that is left is:

$$\sum_{n=6}^{20} n = 6 + 7 + 8 + \cdots + 20$$

The number of terms is $20 - 6 + 1 = 15$

$$\therefore S_{15} = \frac{15}{2}[2(6) + (15-1)(1)]$$
$$= 195$$

There is 195 bottles left after the first 5 rows are sold. (3)

[12]

QUESTION 4: The sum of geometric series:

Given:

$$T_n = ar^{n-1} \qquad S_n = \frac{a(r^n - 1)}{r-1}\ ; r \neq 1 \qquad where \quad r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

4.1 Calculate the sum of the first 15 terms of the series:

$$\frac{2}{3}+2+6+\cdots$$

$$r=\frac{2}{\frac{2}{3}}=\frac{6}{2}=3 \qquad and \qquad a=\frac{2}{3}$$

$$S_n=\frac{a(r^n-1)}{r-1}$$

$$S_{15}=\frac{\frac{2}{3}(3^{15}-1)}{3-1}$$

$$=4\,782\,968{,}67$$ (2)

4.2 Consider the ending series $256+128+64+32+\cdots+0{,}25$

4.2.1 $a=256$

$$r=\frac{128}{256}=\frac{64}{128}=\frac{1}{2}$$

$$\therefore T_n=256\cdot\left(\frac{1}{2}\right)^{n-1}.$$ (1)

4.2.2 $256\cdot\left(\frac{1}{2}\right)^{n-1}=\frac{1}{4}$

$$\therefore\ 2^8\cdot 2^{-n+1}=2^{-2}$$

$$\therefore\quad 8-n+1=-2$$

$$\therefore\quad n=11$$ (1)

4.2.3 The sum of the ending series is:

$$S_n=\frac{a(r^n-1)}{r-1}\ ;r\neq 1$$

$$S_{10}=\frac{256\left(\left(\frac{1}{2}\right)^n-1\right)}{\frac{1}{2}-1}$$

$$=511{,}5$$ (2)

4.3

$$\sum_{k=3}^{m}\frac{1}{16}(2)^k=\frac{1}{16}(2)^3+\frac{1}{16}(2)^4+\frac{1}{16}(2)^5+\cdots+\frac{1}{16}(2)^m$$

4.3.1 $a=\frac{1}{16}(2)^3=\frac{1}{2}$

$r=2$ (1)

4.3.2 The number of terms is:

$m-3+1=m-2$ (1)

4.3.3 Therefore:

$$S_{m-2} = \frac{a(r^{m-2}-1)}{r-1}$$

$$\therefore 127.5 = \frac{0.5(2^{m-2}-1)}{2-1}$$

$$\therefore\ 255 = 2^{m-2} - 1$$

$$\therefore\ 256 = 2^{m-2}$$

$$\therefore\ 2^8 = 2^{m-2}$$

$$\therefore\ 8 = m - 2$$

$$\therefore\ m = 10$$ (3)

[11]

QUESTION 5: The sum to infinite of a convergent geometric series:

5.1 5.1.1 smaller (1)

5.1.2 $(-\mathbf{1} < r < 1)$ (1)

5.2 5.2.1 $T_2 = a \cdot r^1 = \frac{5}{2}$ and $\frac{a}{1-r} = 10$

$\therefore a = \frac{5}{2r}$ $\therefore a = 10(1-r)$

$\therefore \frac{5}{2r} = 10(1-r)$

$\therefore 5 = 20r - 20r^2$

$\therefore 20r^2 - 20r + 5 = 0$

$\therefore (10r - 5)(2r - 1) = 0$

$\therefore r = \frac{5}{10}$ or $r = \frac{1}{2}$

$\therefore r = \frac{1}{2}$ (4)

5.3 Given: $0.\dot{2}\dot{3}$

5.3.1 $0,\dot{2}\dot{3} = 0,23 + 0,0023 + 0,000023 + \cdots$ (1)

5.3.2 $a = 0,23$ and $r = 0,01.$ (1)

5.3.2 The sum to infinite is:

$$S_\infty = \frac{a}{1-r}\ ;\ -1 < r < 1$$

$$S_\infty = \frac{0.23}{1-0.01}$$

$$= \frac{23}{99}$$

(2)

[10]